A Modified JPDA

T. E. Luginbuhl, E. H. Giannopoulos, and P. L. Ainsleigh

Sensors and Sonar Systems Department
Naval Undersea Warfare Center
Newport, RI, 02841–1708 U.S.A.
{luginbuhlte,giannopouloseh,ainsleighpl}@npt.nuwc.navy.mil

Abstract - ***This paper presents a coupled, joint probabilistic data association (JPDA) algorithm for multi-target tracking using a modified version of the standard measurement-to-track assignment model. The mutually exclusive nature of standard JPDA association events precludes any measurement being associated with more than one target in a given event. This constraint is relaxed here to allow a measurement to be assigned to multiple targets. All other JPDA assumptions are retained (i.e., no measurement can be simultaneously associated with target and clutter, and each track can claim at most one measurement). The computational requirements of the resulting algorithm grow linearly with the number of tracks. The recursive estimators for the coupled track means and covariance are derived and presented.***

Keywords: Multi-target tracking, probabilistic data association, estimation

1 Introduction

The joint probabilistic data association (JPDA) algorithm [1–4] for multi-target tracking in clutter has many desirable properties. Prominent among these properties is the fact that the track estimates are obtained from the posterior distribution of the states, thereby providing a more realistic assessment of the uncertainty in the multi-target tracking problem. The single target version, the probabilistic data association filter (PDAF) [5], is a consistent estimator [2, 3, 6] that works well in many different applications. In fact, PDAF is regarded by many to be the baseline for tracking a target in clutter.

The primary drawback of JPDA is the explosive growth in association events that must be considered as the number of targets and detected measurements increase to even moderate levels (≈ 10). Approximations to JPDA that avoid this explosive growth are discussed in references [7–11]. This paper presents a modified JPDA algorithm whose number of association events grow linearly with the number of targets, as opposed to the exponential growth in JPDA. Recently, a new linear complexity multi-target tracking algorithm based on the integrated probabilistic data association algorithm (IPDA) [12] and joint IPDA (JIPDA) [13] was described in references [14–16]. The algorithm in [14–16] is fundamentally different from the algorithm presented here because the assignment assumptions are different. Performance comparisons between the two algorithms are currently being initiated and will be presented at a later time.

The modified JPDA algorithm is developed in the next several sections. The assignment model is presented in Sec. 2. The filtering algorithm that results from this assignment model is discussed at a high level of abstraction in Sec. 3, and these results are specialized to linear Gaussian targets and uniform clutter in Sec. 4. An example containing two crossing targets is then examined in Sec. 5, followed by a short summary and conclusions in Sec. 6.

2 Assignment Model

In the following derivation and discussion, targets are assumed to be close enough together so that a multi-target tracking procedure is required (e.g. the target gates overlap). For widely separated targets or a single target in clutter, the PDAF is employed. For the closely spaced targets, the JPDA algorithm is applied, but with a relaxed set of assignment assumptions. In particular, whereas JPDA does not permit a measurement to be associated with multiple targets in a single event, the new model allows measurements to be assigned to multiple targets (but not target and clutter) by assuming that the measurement-to-track assignments are statistically independent *between targets*. A mutually exclusive and exhaustive set of association events is considered *for each target*.

To establish notation, the event in which the kth measurement is assigned to the jth target track dur-

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ing the nth scan interval, and in which all other measurements are assigned to clutter, is denoted by θ_{jkn}. The vector θ_{jn} represents the collection of association events for the jth target track. Consequently,

$$\theta_{jn} = \bigcup_{k=0}^{K_n} \theta_{jkn} \tag{1}$$

where $\theta_{jkn} \bigcap \theta_{jln} = \emptyset$ for all $k \neq l$,

$$\text{Prob}\left\{\theta_{ikn} \bigcap \theta_{jln}\right\} = \text{Prob}\left\{\theta_{ikn}\right\} \text{Prob}\left\{\theta_{jln}\right\} \tag{2}$$

for $i \neq j$ and K_n represents the number of measurements in scan n. The association events for each target make no explicit allowance for the existence of other targets (unlike the methods described in [14–16]). Multi-target behavior arises due to the mixing that results from the independence of the between-target association events.

The measurements

$$\mathbf{Z}_n = \{\mathrm{z}_{kn}\}_{k=1}^{K_n} \tag{3}$$

($K_n \geq 0$) in scan n are statistically independent when conditioned on the target states, $\mathbf{x}_{jn}$ (for $j = 1, \ldots, J$), and the association events, θ_{jn}. Furthermore, the target states are independent of the association events. Stacking the state vectors from all targets into a single vector defines the combined state vector

$$\mathbf{X}_n = \left[\mathbf{x}_{1n}^{\mathrm{T}}, \ldots, \mathbf{x}_{Jn}^{\mathrm{T}}\right]^{\mathrm{T}} \tag{4}$$

whose evolution from scan to scan is governed by a set of independent first order Markov processes, such that

$$p(\mathbf{X}_n|\mathbf{X}_{n-1}) = \prod_{j=1}^{J} p(\mathbf{x}_{jn}|\mathbf{x}_{jn-1}). \tag{5}$$

Here, J denotes the number of targets in the observation space with overlapping target gates. While the measurements are conditionally independent when given the combined state vector, the individual target state estimates are not necessarily independent when target gates overlap. Dependent target state estimates are said to be *coupled*, and they are estimated jointly [3].

The posterior event probabilities for the modified algorithm are obtained via the usual application of Bayes theorem. The measurement likelihood for event θ_{jkn}, in which the kth measurement goes to the jth target, is given by

$$p(\mathbf{Z}_n|\mathbf{x}_{jn}, \theta_{jkn}, j) = p(\mathrm{z}_{kn}|\mathbf{x}_{jn}) \prod_{\substack{r=1 \\ r\neq k}}^{K_n} p_c(\mathrm{z}_{rn}) \tag{6}$$

where $p_c(\cdot)$ denotes the likelihood of a clutter measurement. For the event that assigns all measurements to clutter (i.e., $k = 0$), the measurement likelihood is given by

$$p(\mathbf{Z}_n|\theta_{j0n}, j) = \prod_{k=1}^{K_n} p_c(\mathrm{z}_{kn}). \tag{7}$$

The joint likelihood of the measurements conditioned on the state for target j is obtained by marginalizing over the association events for target j to obtain

$$p(\mathbf{Z}_n|\mathbf{x}_{jn}, j) = \sum_{k=0}^{K_n} p(\mathbf{Z}_n|\mathbf{x}_{jn}, \theta_{jkn}, j) \text{ Prob}\left\{\theta_{jkn}\right\}. \tag{8}$$

This expression is the conditional measurement likelihood for the (single-target) PDAF. Similarly, the likelihood of the measurements in scan n, conditioned on all the target states, is obtained by marginalizing over the targets as

$$p(\mathbf{Z}_n|\mathbf{X}_n) = \sum_{j=1}^{J} \text{Prob}\left\{j\right\} \; p(\mathbf{Z}_n|\mathbf{x}_{jn}, j) \tag{9}$$

where Prob $\{j\}$ represents the prior probability of target j.

3 Posterior Estimates

The estimates of the target states that minimize the posterior mean square error and the corresponding covariance matrices at each scan are obtained by taking expectations with respect to the conditional target state densities. The process of computing these expectations is broken into two steps consisting of a time update and a measurement update as described in [17] and [18]. Just enough of the theory is repeated in this section to introduce needed notation.

The starting point for developing state estimators is the joint likelihood function

$$\begin{aligned} &p(\mathbf{Z}_n, \mathbf{X}_n, \mathbf{X}_{n-1}|\mathcal{Z}_{n-1}) \\ &\quad = p(\mathbf{Z}_n|\mathbf{X}_n)\, p(\mathbf{X}_n, \mathbf{X}_{n-1}|\mathcal{Z}_{n-1}) \\ &\quad = p(\mathbf{Z}_n|\mathbf{X}_n)\, p(\mathbf{X}_n|\mathbf{X}_{n-1})\, p(\mathbf{X}_{n-1}|\mathcal{Z}_{n-1}) \end{aligned} \tag{10}$$

where $\mathcal{Z}_{n-1}$ represents the collection of all the measurements from the starting scan through and including scan $n-1$. Given the (prior) PDF of the previous state estimates, $p(\mathbf{X}_{n-1}|\mathcal{Z}_{n-1})$, the time update forms

$$p(\mathbf{X}_n|\mathcal{Z}_{n-1}) = \int d\mathbf{X}_{n-1}\, p(\mathbf{X}_n, \mathbf{X}_{n-1}|\mathcal{Z}_{n-1}). \tag{11}$$

The measurement update applies Bayes rule to obtain

$$p(\mathbf{X}_n|\mathcal{Z}_n) = p(\mathbf{X}_n|\mathbf{Z}_n, \mathcal{Z}_{n-1}) = \frac{p(\mathbf{Z}_n|\mathbf{X}_n)p(\mathbf{X}_n|\mathcal{Z}_{n-1})}{p(\mathbf{Z}_n|\mathcal{Z}_{n-1})}. \quad (12)$$

The denominator term $p(\mathbf{Z}_n|\mathcal{Z}_{n-1})$ is the combined innovation likelihood:

$$p(\mathbf{Z}_n|\mathcal{Z}_{n-1}) = \int d\mathbf{X}_n \, p(\mathbf{Z}_n, \mathbf{X}_n|\mathcal{Z}_{n-1}). \quad (13)$$

In practical applications, $p(\mathbf{Z}_n|\mathcal{Z}_{n-1})$ is based on a truncated density function such that only measurements near the predicted target measurement are considered by the algorithm. This procedure is referred to as gating or measurement validation [2–4, 19].

Given the posterior state density, the state is estimated as

$$\hat{\mathbf{X}}_{n|n} = \int d\mathbf{X}_n \, \mathbf{X}_n \, p(\mathbf{X}_n|\mathcal{Z}_n). \quad (14)$$

Defining the outer-product function $g(\mathbf{X}) = \mathbf{X}\mathbf{X}^{\mathrm{T}}$, the posterior covariance matrix of the combined target state vector is given by

$$\mathcal{P}_{n|n} = \int d\mathbf{X}_n \, g(\mathbf{X}_n - \hat{\mathbf{X}}_{n|n}) \, p(\mathbf{X}_n|\mathcal{Z}_n). \quad (15)$$

4 Linear Gaussian Targets in Poisson Clutter

This section develops the time and measurement updates for linear-Gaussian targets and Poisson clutter. Target measurements are detected with a probability of P_d, and clutter measurements are uniformly distributed in the observation space. The modified JPDA algorithm considers an observation space that is the union of the overlapping target gates. Denoting by $\mathbf{V}$ the volume in the union of the gates, the clutter density in this observation region is

$$p_c(\mathrm{z}_{kn}) = \mathbf{V}^{-1}. \quad (16)$$

The number of clutter measurements is modeled by a Poisson distribution with parameter $\lambda\mathbf{V}$, where λ is the average number per unit volume (spatial density). The number of measurements, K_n, in a scan is equal to the sum of the number of target measurements and the number of clutter measurements which are statistically independent.

The number of target measurements in any given scan is between zero (no target produces a measurement) and J (all targets produce a measurement) and has a binomial distribution with parameter P_d. The distribution of K_n is obtained by convolving the binomial distribution for the number of target measurements with the Poisson distribution for the number of clutter measurements, which gives

$$\mathrm{Prob}\{K_n\} = \mathrm{e}^{-\lambda V} \sum_{l=0}^{J'} \binom{J'}{l} P_d^l (1-P_d)^{J'-l} \frac{(\lambda V)^{K_n - l}}{(K_n - l)!} \quad (17)$$

where $J' = \min(K_n, J)$. Using Eq. (17), and following the derivation in Appendix D.4 in [2], the prior probability for the event in which the kth measurement is assigned to the jth target (and all other measurements to clutter) is given by

$$\mathrm{Prob}\{\theta_{jkn}\} = \mathrm{Prob}\{K_n\}^{-1} \begin{cases} f_1(K_n)/K_n & \text{for } 1 \le k \le K_n \\ f_0(K_n) & \text{for } k = 0 \end{cases} \quad (18)$$

where

$$f_1(K_n) = \mathrm{e}^{-\lambda V} \sum_{l=1}^{J'} \binom{J'}{l} P_d^l (1-P_d)^{J'-l} \frac{(\lambda V)^{K_n - l}}{(K_n - l)!} \quad (19)$$

and

$$f_0(K_n) = \mathrm{e}^{-\lambda V} (1-P_d)^{J'} \frac{(\lambda V)^{K_n}}{K_n!}. \quad (20)$$

When only one target is present Eq. (18) reduces to the event prior probability given for the PDAF in [2, 3].

The posterior event probabilities are obtained by including the appropriate innovation likelihood terms. Under linear Gaussian assumptions, the likelihood of a target measurement equals

$$p(\mathrm{z}_{kn}|\mathbf{x}_{jn}) = \mathcal{N}(\mathrm{z}_{kn}; \mathbf{B}_{jn}\mathbf{x}_{jn}, \mathbf{R}_{jn}) = \mathcal{N}\left(\mathrm{z}_{kn}; \tilde{\mathbf{B}}_{jn}\mathbf{X}_n, \mathbf{R}_{jn}\right) \quad (21)$$

where $\mathcal{N}(x; \mu, \sigma)$ denotes the Gaussian density with mean μ and covariance σ and

$$\tilde{\mathbf{B}}_{jn} = [0 \cdots 0 \, \mathbf{B}_{jn} \, 0 \cdots 0]. \quad (22)$$

The density function for a target's current state, given the previous state, is defined as

$$p(\mathbf{x}_{jn}|\mathbf{x}_{jn-1}) = \mathcal{N}\left(\mathbf{x}_{jn}; \mathbf{A}_{jn}\mathbf{x}_{jn-1}, \mathbf{Q}_{jn}\right). \quad (23)$$

The joint density of all target states at scan n, conditioned on all the data up to and including scan n, is also Gaussian and is given by

$$p(\mathbf{X}_n|\mathcal{Z}_n) = \mathcal{N}\left(\mathbf{X}_n; \hat{\mathbf{X}}_{n|n}, \mathcal{P}_{n|n}\right). \quad (24)$$

Because the covariance matrix $\mathcal{P}_{n|n}$ is not block diagonal (as will be shown subsequently) the target state estimates are coupled.

4.1 Time Update

For linear Gaussian targets, Eq. (5) is equivalent to

$$p(\mathbf{X}_n|\mathbf{X}_{n-1}) = \mathcal{N}\left(\mathbf{X}_n; \mathcal{A}_n\mathbf{X}_{n-1}, \mathcal{Q}_n\right) \quad (25)$$

where $\mathcal{A}_n = \operatorname{diag}\{\mathbf{A}_{jn}\}$ is the block diagonal matrix of the target state-feedback matrices and $\mathcal{Q}_n = \operatorname{diag}\{\mathbf{Q}_{jn}\}$. The time update therefore amounts to marginalizing the previous state from the joint Gaussian density

$$\begin{aligned} p(\mathbf{X}_n, \mathbf{X}_{n-1}|\mathcal{Z}_{n-1}) & \\ &= p(\mathbf{X}_n|\mathbf{X}_{n-1})\, p(\mathbf{X}_{n-1}|\mathcal{Z}_{n-1}) \\ &= \mathcal{N}\left(\mathbf{X}_n; \mathcal{A}_n\mathbf{X}_{n-1}, \mathcal{Q}_n\right) \\ &\quad \times \mathcal{N}\left(\mathbf{X}_{n-1}; \hat{\mathbf{X}}_{n-1|n-1}, \mathcal{P}_{n-1|n-1}\right). \end{aligned} \quad (26)$$

For this type of Gaussian product, the dependency reversal from Bayes theorem is easily achieved by applying the Gaussian refactorization lemma (GRL) [20], which directly yields the expression

$$\begin{aligned} p(\mathbf{X}_n, \mathbf{X}_{n-1}|\mathcal{Z}_{n-1}) & \\ &= p(\mathbf{X}_n|\mathcal{Z}_{n-1})\, p(\mathbf{X}_{n-1}|\mathbf{X}_n, \mathcal{Z}_{n-1}) \\ &= \mathcal{N}\left(\mathbf{X}_n; \hat{\mathbf{X}}_{n|n-1}, \mathcal{P}_{n|n-1}\right) \mathcal{N}\left(\mathbf{X}_{n-1}; \mathbf{U}_n, \Lambda_n\right) \end{aligned} \quad (27)$$

where

$$\hat{\mathbf{X}}_{n|n-1} = \mathcal{A}_n\, \hat{\mathbf{X}}_{n-1|n-1} \quad (28)$$

$$\mathcal{P}_{n|n-1} = \mathcal{Q}_n + \mathcal{A}_n\, \mathcal{P}_{n-1|n-1}\, \mathcal{A}_n^{\mathrm{T}} \quad (29)$$

$$\mathbf{U}_n = \mathbf{W}_n\, \hat{\mathbf{X}}_{n-1|n-1} + \mathbf{H}_n\, \mathbf{X}_n \quad (30)$$

$$\Lambda_n = \mathbf{W}_n\, \mathcal{P}_{n-1|n-1} \quad (31)$$

$$\mathbf{W}_n = \mathbf{I} - \mathbf{H}_n\, \mathcal{A}_n \quad (32)$$

and

$$\mathbf{H}_n = \mathcal{P}_{n-1|n-1}\, \mathcal{A}_n^{\mathrm{T}}\, \mathcal{P}_{n|n-1}^{-1}. \quad (33)$$

With this dependency reversal, integrating the right side of Eq. (27) with respect to $\mathbf{X}_{n-1}$ is trivial, giving

$$p(\mathbf{X}_n|\mathcal{Z}_{n-1}) = \mathcal{N}\left(\mathbf{X}_n; \hat{\mathbf{X}}_{n|n-1}, \mathcal{P}_{n|n-1}\right) \quad (34)$$

which is the familiar predicted state density. Since the covariance matrix, $\mathcal{P}_{n|n-1}$, of the combined state estimate $\hat{\mathbf{X}}_{n|n-1}$ is *not*, in general, block diagonal, the predicted target state densities are not statistically independent.

4.2 Innovation Likelihood

The fact that the predicted target states are coupled complicates the derivation of $p(\mathbf{Z}_n|\mathcal{Z}_{n-1})$ to some degree. To address this problem, the combined state output matrix $\tilde{\mathbf{B}}_{jn}$ was introduced in Eq. (21). By using $\tilde{\mathbf{B}}_{jn}\mathbf{X}_n$ instead of $\mathbf{B}_{jn}\mathbf{x}_{jn}$, the GRL can be applied directly to (34) such that marginalization with respect to the combined target state is straightforward.

Since the clutter terms in Eq. (6) do not depend on the state vector for target j, the product $p(\mathbf{Z}_n|\mathbf{x}_{jn}, \theta_{jkn})\, p(\mathbf{X}_n|\mathcal{Z}_{n-1})$ is marginalized by applying the GRL to the two Gaussian components, integrating $\mathbf{X}_n$ from the result, and then multiplying by the clutter terms after the fact. The Gaussian components are re-factorized as

$$\begin{aligned} &\mathcal{N}\left(\mathrm{z}_{kn}; \tilde{\mathbf{B}}_{jn}\mathbf{X}_n, \mathbf{R}_{jn}\right) \mathcal{N}\left(\mathbf{X}_n; \hat{\mathbf{X}}_{n|n-1}, \mathcal{P}_{n|n-1}\right) \\ &\quad = \mathcal{N}\left(\mathrm{z}_{kn}; \hat{\mathrm{z}}_{jn}, \mathbf{S}_{jn}\right) \mathcal{N}\left(\mathbf{X}_n; \mathbf{X}_{jk,n|n}, \mathcal{P}_{jk,n|n}\right) \end{aligned} \quad (35)$$

where

$$\hat{\mathrm{z}}_{jn} = \tilde{\mathbf{B}}_{jn}\, \hat{\mathbf{X}}_{n|n-1} \quad (36)$$

$$\mathbf{S}_{jn} = \mathbf{R}_{jn} + \tilde{\mathbf{B}}_{jn}\, \mathcal{P}_{n|n-1}\, \tilde{\mathbf{B}}_{jn}^{\mathrm{T}} \quad (37)$$

$$\mathbf{X}_{jk,n|n} = \mathbf{D}_{jn}\, \hat{\mathbf{X}}_{n|n-1} + \mathbf{G}_{jn}\, \mathrm{z}_{kn} \quad (38)$$

$$\mathcal{P}_{jk,n|n} = \mathbf{D}_{jn}\, \mathcal{P}_{n|n-1} \quad (39)$$

$$\mathbf{D}_{jn} = \mathbf{I} - \mathbf{G}_{jn}\, \tilde{\mathbf{B}}_{jn} \quad (40)$$

and

$$\mathbf{G}_{jn} = \mathcal{P}_{n|n-1}\, \tilde{\mathbf{B}}_{jn}^{\mathrm{T}}\, \mathbf{S}_{jn}^{-1}. \quad (41)$$

Integrating Eq. (35) with respect to $\mathbf{X}_n$ gives

$$\begin{aligned} p(\mathrm{z}_{kn}|\hat{\mathrm{z}}_{jn}) &= \int d\mathbf{X}_n\, p(\mathrm{z}_{kn}|\mathbf{x}_{jn})\, p(\mathbf{X}_n|\hat{\mathbf{X}}_{n|n-1}) \\ &= \mathcal{N}\left(\mathrm{z}_{kn}; \hat{\mathrm{z}}_{jn}, \mathbf{S}_{jn}\right). \end{aligned} \quad (42)$$

Inclusion of the clutter terms then gives conditional likelihood as

$$p(\mathbf{Z}_n|\theta_{jkn}, \mathcal{Z}_{n-1}) = \mathbf{V}^{1-K_n}\, \mathcal{N}\left(\mathrm{z}_{kn}; \hat{\mathrm{z}}_{jn}, \mathbf{S}_{jn}\right) \quad (43)$$

for $1 \leq k \leq K_n$. For $k = 0$, where all measurements are presumed to be clutter, this expression reduces to

$$p(\mathbf{Z}_n|\theta_{0kn}, \mathcal{Z}_{n-1}) = \mathbf{V}^{-K_n}. \quad (44)$$

The overall innovation likelihood is obtained by nested application of the total probability theorem as

$$\begin{aligned} p(\mathbf{Z}_n|\mathcal{Z}_{n-1}) &= \sum_{j=1}^{J} \operatorname{Prob}\{\, j\} \\ &\quad \times \sum_{k=0}^{K_n} \operatorname{Prob}\{\theta_{jkn}\}\; p(\mathbf{Z}_n|\theta_{jkn}, \mathcal{Z}_{n-1}). \end{aligned} \quad (45)$$

4.3 Measurement Update

Substituting equations (34) and (45) into Eq. (12) yields the posterior density of the combined target state as

$$p(\mathbf{X}_n|\mathcal{Z}_{n-1}) = \frac{p(\mathbf{Z}_n|\mathbf{X}_n)\,\mathcal{N}\left(\mathbf{X}_n;\hat{\mathbf{X}}_{n|n-1},\mathcal{P}_{n|n-1}\right)}{p(\mathbf{Z}_n|\mathcal{Z}_{n-1})}. \tag{46}$$

The state estimates are obtained by taking expectations with respect to this conditional density. In computing the conditional mean, there are two types of expectations that need to be evaluated. For each measurement-to-target association hypothesis, the expected value of $\mathbf{X}_n$ is given by

$$\begin{aligned}\mathbf{X}_{jk,n|n} &= p(\mathbf{Z}_n|\mathcal{Z}_{n-1})^{-1}\\ &\times \int d\mathbf{X}_n\;\mathbf{X}_n\;p(\mathbf{Z}_n|\mathbf{x}_{jn},\theta_{jkn})\;p(\mathbf{X}_n|\hat{\mathbf{X}}_{n|n-1}).\end{aligned} \tag{47}$$

where $\mathbf{X}_{j0,n|n} = \hat{\mathbf{X}}_{n|n-1}$. The posterior mean of the combined target state is therefore

$$\begin{aligned}\hat{\mathbf{X}}_{n|n} &= \sum_{j=1}^{J}\sum_{k=0}^{K_n}\psi_{jkn}\,\mathbf{X}_{jk,n|n}\\ &= \hat{\mathbf{X}}_{n|n-1} + \sum_{j=1}^{J}\mathbf{G}_{jn}\sum_{k=1}^{K_n}\psi_{jkn}\;(\mathrm{z}_{nk}-\hat{\mathrm{z}}_{jn})\end{aligned} \tag{48}$$

where

$$\psi_{jkn} = \frac{\mathrm{Prob}\,\{j\}\;\mathrm{Prob}\,\{\theta_{jkn}\}\;p(\mathbf{Z}_n|\theta_{jkn},\mathcal{Z}_{n-1})}{p(\mathbf{Z}_n|\mathcal{Z}_{n-1})}. \tag{49}$$

The combined state covariance is obtained following similar developments. Recalling the definition $g(\mathbf{X}) = \mathbf{X}\mathbf{X}^{\mathrm{T}}$ and noting that

$$g\left(\mathbf{X}_n-\hat{\mathbf{X}}_{n|n}\right) = g\left(\mathbf{X}_n-\mathbf{X}_{jk,n|n}+\mathbf{X}_{jk,n|n}-\hat{\mathbf{X}}_{n|n}\right)$$

the covariance matrix for each measurement-to-target association event is defined by

$$\begin{aligned}&\mathcal{P}_{jk,n|n} + g\left(\mathbf{X}_{jk,n|n}-\hat{\mathbf{X}}_{n|n}\right)\\ &\quad= p(\mathbf{Z}_n|\mathcal{Z}_{n-1})^{-1}\int d\mathbf{X}_n\;g\left(\mathbf{X}_n-\hat{\mathbf{X}}_{n|n}\right)\\ &\qquad\times\;p(\mathbf{Z}_n|\mathbf{x}_{jn},\theta_{jkn})\,p(\mathbf{X}_n|\hat{\mathbf{X}}_{n|n-1}).\end{aligned} \tag{50}$$

where $\mathcal{P}_{j0,n|n} = \mathcal{P}_{n|n-1}$. Combining terms then yields

$$\begin{aligned}&\mathcal{P}_{n|n} =\\ &\sum_{j=1}^{J}\sum_{k=0}^{K_n}\psi_{jkn}\left[\mathcal{P}_{jk,n|n}+g\left(\mathbf{X}_{jk,n|n}-\hat{\mathbf{X}}_{n|n}\right)\right].\end{aligned} \tag{51}$$

Finally, substituting Eqs. (39)–(41) for $\mathcal{P}_{n|n}$ and simplifying gives

$$\begin{aligned}\mathcal{P}_{n|n} &= (\mathbf{I}-\mathbf{G}_n)\mathcal{P}_{n|n-1}\\ &\quad+\sum_{j=1}^{J}\sum_{k=0}^{K_n}\psi_{jkn}\;g\left(\mathbf{X}_{jk,n|n}-\hat{\mathbf{X}}_{n|n}\right)\end{aligned} \tag{52}$$

where

$$\mathbf{G}_n = \sum_{j=1}^{J}\left(\sum_{k=1}^{K_n}\psi_{jkn}\right)\mathbf{G}_{jn}\,\tilde{\mathbf{B}}_{jn}. \tag{53}$$

An alternative method of computing $\hat{\mathbf{X}}_{n|n}$ and $\mathcal{P}_{n|n}$ is obtained by breaking equations (48) and (52) into two convex sums instead of one. The first convex sum is over measurements and is essentially a PDAF step. The second convex sum mixes the PDAF estimates. Approximating the weights used in this mixing between tracks (second convex sum) by the prior target probabilities yields a far more stable algorithm. To begin the PDAF step, let

$$\beta_{jn} = \sum_{k=0}^{K_n}p(\mathbf{Z}_n|\theta_{jkn},\mathcal{Z}_{n-1})\,\mathrm{Prob}\,\{\theta_{jkn}\} \tag{54}$$

and

$$\beta_{jkn} = \beta_{jn}^{-1}\,p(\mathbf{Z}_n|\theta_{jkn},\mathcal{Z}_{n-1})\,\mathrm{Prob}\,\{\theta_{jkn}\}\,. \tag{55}$$

Then, the contribution to the combined state vector from target j equals

$$\begin{aligned}\hat{\mathbf{X}}_{j,n|n} &= \sum_{k=0}^{K_n}\beta_{jkn}X_{jk,n|n}\\ &= \hat{\mathbf{X}}_{n|n-1}+G_{jn}\sum_{k=1}^{K_n}\beta_{jkn}\;\left(\mathrm{z}_{kn}-\hat{\mathrm{z}}_{n|n-1}\right)\end{aligned} \tag{56}$$

which is the same as the PDAF's target state estimate except that, in this case, it is the contribution of target j to the combined state vector. To compute $\mathcal{P}_{n|n}$, the average distances from $\mathbf{X}_{jk,n|n}$ to $\mathbf{X}_{j,n|n}$ and $\mathbf{X}_{j,n|n}$ to $\mathbf{X}_{n|n-1}$ are required:

$$\mathbf{Y}_{j,n|n} = \sum_{k=1}^{K_n}\beta_{jkn}\;\left(\mathbf{X}_{jk,n|n}-\hat{\mathbf{X}}_{j,n|n}\right) \tag{57}$$

and

$$\mathbf{Y}_{j,n|n-1} = \beta_{j0n}\;\left(\hat{\mathbf{X}}_{j,n|n}-\hat{\mathbf{X}}_{n|n-1}\right). \tag{58}$$

The covariance contribution from target j is given by

$$\begin{aligned}\mathcal{P}_{j,n|n} &= \beta_{j0n}\left[\mathcal{P}_{n|n-1}+g\left(\hat{\mathbf{X}}_{n|n-1}-\hat{\mathbf{X}}_{j,n|n}\right)\right]\\ &\quad+\sum_{k=1}^{K_n}\beta_{jkn}\left[\mathcal{P}_{jk,n|n}+g\left(\mathbf{X}_{jk,n|n}-\hat{\mathbf{X}}_{j,n|n}\right)\right]\end{aligned} \tag{59}$$

which is the same form as the PDAF covariance estimate.

To form the single combined state estimate and covariance, we introduce the posterior probability of target j, given by

$$\omega_{jn} = \frac{\text{Prob}\{j\}\, \beta_{jn}}{p(\mathbf{Z}_n|\mathcal{Z}_{n-1})} \tag{60}$$

and observe that

$$\sum_{j=1}^{J} \omega_{jn} = 1. \tag{61}$$

Because ω_{jn} is the posterior probability of target j, and because $p(\mathbf{Z}_n|\mathcal{Z}_{n-1})$ is the average value of the β_{jn}'s (see equations (45) and (54)), it follows that

$$\omega_{jn} \approx \text{Prob}\{j\}\,. \tag{62}$$

Using this approximation for ω_{jn} yields more stable estimates because it prevents one or more targets from dominating the mixing across targets, which can cause the covariance estimates for the remaining targets to blow up over time.

Using Eq. (56), the estimate of the combined target state vector from Eq. (48) is given by

$$\hat{\mathbf{X}}_{n|n} = \sum_{j=1}^{J} \omega_{jn} \hat{\mathbf{X}}_{j,n|n}. \tag{63}$$

Similarly, Eq. (52) now may be expressed as

$$\begin{aligned}\mathcal{P}_{n|n} = \sum_{j=1}^{J} \omega_{jn} \Big[\mathcal{P}_{j,n|n} + g\left(\hat{\mathbf{X}}_{j,n|n} - \hat{\mathbf{X}}_{n|n}\right) \\ + \mathbf{Y}_{j,n|n}\left(\hat{\mathbf{X}}_{j,n|n} - \hat{\mathbf{X}}_{n|n}\right)^{\mathrm{T}} + \left(\hat{\mathbf{X}}_{j,n|n} - \hat{\mathbf{X}}_{n|n}\right)\mathbf{Y}^{\mathrm{T}}_{j,n|n} \\ + \mathbf{Y}_{j,n|n-1}\left(\hat{\mathbf{X}}_{j,n|n} - \hat{\mathbf{X}}_{n|n}\right)^{\mathrm{T}} \\ + \left(\hat{\mathbf{X}}_{j,n|n} - \hat{\mathbf{X}}_{n|n}\right)\mathbf{Y}^{\mathrm{T}}_{j,n|n-1}. \Big]\end{aligned} \tag{64}$$

5 Example

A two target crossing scenario was used to compare the performance of the Modified JPDA algorithm against single target PDAF and a multiple target JPDA. Performance criteria include the number of lost tracks, the Normalized Estimation Error Squared (NEES), positional error, and algorithm complexity. A lost track is defined as any track whose NEES value exceeds twenty during any update. One thousand Monte Carlo trials were used to generate the results. The scenario parameters used for the simulation are given in table 1,

Update Rate	1 Sec
Number of Updates	65
Target 1 Course	88 Deg
Target 2 Course	92 Deg
Target 1 Init Pos	(0, 1200) m
Target 2 Init Pos	(0, 2000)m
Target 1 Speed	500 m/s
Target 2 Speed	500 m/s
Process Noise Sigma	0.01
Measurement Noise Sigma	75
λ	1 per km^2
P_d	0.99
P_G	0.99

Table 1: Crossing tracks scenario parameters

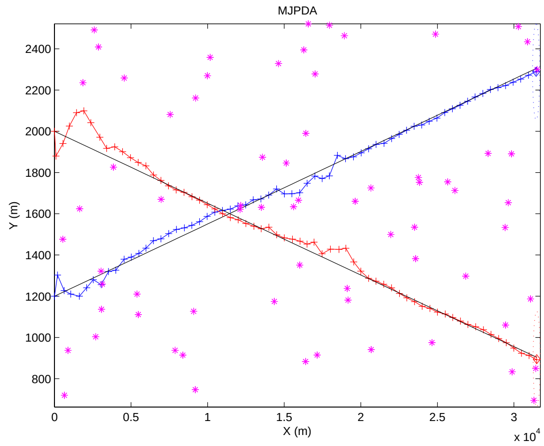

Figure 1: Sample scenario run with MJPDA results

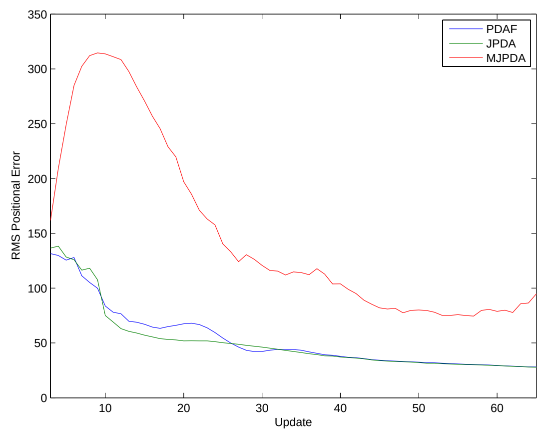

Figure 2: RMS positional error curves for PDAF, JPDA and MJPDA

Alg.	% Lost Tracks	NEES	Rel. Pos. Error	Complexity
PDAF	18.8	Consistent	1	Linear
JPDA	8.0	Consistent	1	Exponential
MJPDA	13.0	Mostly consistent	3	Linear

Table 2: Performance summary

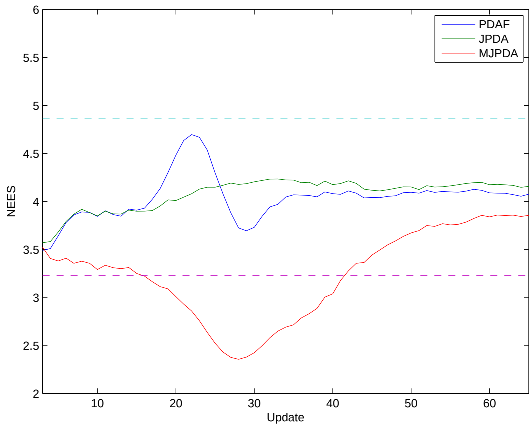

Figure 3: NEES curves for PDAF, JPDA and MJPDA

and an example showing results for MJPDA appears in figure 1. The results from the runs are summarized in table 2 and figures 2 and 3.

This example indicates that the modified JPDA algorithm's lost track performance is better than that of independent PDAFs, but is degraded relative to JPDA. Furthermore, its covariance estimates are too large during the cross, making the NEES measure too small in that region, and the position errors are larger than PDAF and JPDA. The degraded performance characteristics are believed to result from splitting all the measurements in the union of two intersecting gates between the two targets. As a result, the measurements have a reduced impact on the updates to the target states in Eqs. (48) and (63) and the target state covariance matrices in Eqs. (52) and (64). The predicted state and covariance are therefore more influential, causing the state covariance to grow too large, which, in turn, causes the NEES plot for MJPDA to fall below the lower threshold in figure 3. This problem may be addressed by splitting between targets only those measurements that fall within the *intersection* (as opposed to the union) of the target gates. An investigation of that approach is ongoing.

6 Summary

This paper presents a modified, coupled JPDA algorithm that achieves linear computational complexity by relaxing the requirement that a measurement can only be assigned to one target. Instead, this new algorithm assumes that the measurement to target assignment events for target are statistically independent, allowing a given measurement to be assigned to multiple targets. In this single target case, the new algorithm reduces to the PDAF.

The intent of this work is to develop a consistent estimator whose performance is near to that of JPDA but whose computational growth is linear in the number of targets and measurements (instead of exponential). Future work will include an examination of the MJPDA algorithm in which only those measurements in the intersection of gates is split between targets (as discussed in the previous section) and a performance comparison between MJPDA and the IPDA-based algorithm [14–16].

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